Exam Geometry WIMTK-08

January 25, 2017

Note: This exam consists of **four** problems. Usage of Do Carmo's textbook is allowed. Give a precise reference to the theory you use for solving the problems.

Problem 1 (3+5+5+5+7=25 pts.)
Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be the curve given by

$$\alpha(s) = \big(\tfrac{1}{\sqrt{2}}\cos s, \tfrac{1}{2}(-s + \sin s), \tfrac{1}{2}(s + \sin s)\big)$$

The curvature and torsion at $\alpha(s)$ are denoted by $k(s)$ and $\tau(s)$, respectively.

1. Show that the curve is parameterized by arc length.
2. Determine the Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ of the curve at an arbitrary point $\alpha(s)$.
3. Show that the curvature k and the torsion τ of the curve are constant by computing their values.
4. Let $\mathbf{v}(s)$ be the vector given by $\mathbf{v}(s) = -\tau\,\mathbf{t}(s) + k\,\mathbf{b}(s)$. Prove that $\mathbf{v}$ is a constant vector.
5. Denote the constant value of $\mathbf{v}(s)$ by $\mathbf{v}_0$. Show that the curve α lies on a circular cylinder the axis of which is the line through the origin with direction vector $\mathbf{v}_0$. (Hint: prove that the distance of $\alpha(s)$ to this line is constant.)

Problem 2 (8+12=20 pts.)
Let S be the surface in $\mathbb{R}^3$ with equation

$$z = \tfrac{1}{2}ax^2 + \tfrac{1}{2}by^2,$$

with $a \neq b$.

1. Determine the Gaussian curvature at a point (x, y, z) of S.
2. Let V be a plane through the z-axis. Prove that the curve $S \cap V$ is a line of curvature of S iff V is the xz-plane or V is the yz-plane.

Problem 3 (12+8 = 20 pts.)
Let S be the torus of revolution obtained by rotating the circle with equation

$$(x-a)^2 + z^2 = r^2, \qquad y = 0,$$

about the z-axis. Here a and r are positive constants, with $a > r > 0$. The parallels through the points $(a+r, 0, 0)$, $(a-r, 0, 0)$ and $(a, r, 0)$ are called the *maximum parallel*, the *minimum parallel*, and the *upper parallel*, respectively.

1. Check which of these parallels is
   (a) a geodesic;
   (b) a line of curvature;
   (c) an asymptotic curve.
2. Determine the geodesic curvature of the upper parallel of the torus.

Remark: this problem can be solved using geometric arguments. In other words, you hardly need to do computations.

Problem 4 (8 + 7 + 10 = 25 pts.)
Let $\mathbf{x} : U \to \mathbb{R}^3$ be a regular parametrization, where U is an open subset of $\mathbb{R}^2$. As usual, the coefficients of the first fundamental form of $\mathbf{x}$ are denoted by E, F and G, respectively. The parametrization is such that

$$E(u,v) = 1 \quad \text{and} \quad F(u,v) = 0,$$

for all $(u,v) \in U$.

1. Prove that the curve $\alpha(s) = \mathbf{x}(s, v_0)$ is a geodesic, where s ranges over an interval I such that $(s, v_0) \in U$ for $s \in I$. (In other words: prove that all u-curves are geodesics.)
2. Let I be an interval and $v : I \to \mathbb{R}$ be a C^2-function such that $(u_0, v(s)) \in U$ for $s \in I$. Prove that the curve $\beta(s) = \mathbf{x}(u_0, v(s))$ has constant non-zero speed if and only if

   $$v' \text{ is nowhere zero and } 2Gv'' + G_v(v')^2 = 0, \quad \text{for } s \in I.$$

   Here G and G_v are evaluated at $(u_0, v(s))$, whereas v' and v'' are evaluated at s.
3. Prove that a regular curve β with constant speed as in part 2 is a geodesic, for all u_0 for which it is defined, if and only if $G(u,v)$ does not depend on u. (In other words, prove that all v-curves are geodesics up to reparametrization if and only if G does not depend on u.)

Solutions

Problem 1.

1. A straightforward computation shows

$$\alpha'(s) = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin s \\ \frac{1}{2}(-1 + \cos s) \\ \frac{1}{2}(1 + \cos s) \end{pmatrix},$$

so $|\alpha'(s)| = 1$.

2. The first frame vector is $\mathbf{t}(s) = \alpha'(s)$, which has been computed in Part 1. Since the curve has unit speed,

$$\mathbf{n}(s) = \frac{\mathbf{t}'(s)}{|\mathbf{t}'(s)|} = \begin{pmatrix} -\cos s \\ -\frac{1}{\sqrt{2}} \sin s \\ -\frac{1}{\sqrt{2}} \sin s \end{pmatrix}.$$

Finally,

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} \sin s \\ \frac{1}{2}(-1 - \cos s) \\ \frac{1}{2}(1 - \cos s) \end{pmatrix}.$$

3. In view of the Frenet formulas on page 19 of Do Carmo's book we get

$$k = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle = \langle \begin{pmatrix} -\frac{1}{\sqrt{2}} \cos s \\ -\frac{1}{2} \sin s \\ -\frac{1}{2} \sin s \end{pmatrix}, \begin{pmatrix} -\cos s \\ -\frac{1}{\sqrt{2}} \sin s \\ -\frac{1}{\sqrt{2}} \sin s \end{pmatrix} \rangle = \frac{1}{\sqrt{2}}.$$

Similarly,

$$\tau = \langle \mathbf{b}'(s), \mathbf{n}(s) \rangle = -\frac{1}{\sqrt{2}}.$$

4. Using the Frenet formulas again we get

$$\mathbf{v}'(s) = -k\tau\, \mathbf{n}(s) + k\tau\, \mathbf{n}(s) = \mathbf{0}.$$

Therefore, $\mathbf{v}(s)$ is constant. The value of $\mathbf{v}(s)$ can also be determined explicitly, as we do in the solution of Part 5.

5. We have to prove that the distance from $\alpha(s)$ to its projection $\beta(s)$ onto the line through the origin with direction vector $\mathbf{v}_0$ is constant. To this end we first determine $\mathbf{v}_0$:

$$\mathbf{v}_0 = \frac{1}{\sqrt{2}} (\mathbf{t}(s) + \mathbf{b}(s)) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Since $\mathbf{v}_0$ is a unit vector, we see that

$$\beta(s) = \langle \alpha(s), \mathbf{v}_0 \rangle \, \mathbf{v}_0 = \begin{pmatrix} 0 \\ -\frac{1}{2}s \\ \frac{1}{2}s \end{pmatrix}.$$

Therefore,

$$\beta(s) - \alpha(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos s \\ \frac{1}{2} \sin s \\ \frac{1}{2} \sin s \end{pmatrix},$$

so $|\beta(s) - \alpha(s)| = \frac{1}{\sqrt{2}}$, for all s. This concludes the proof of the statement.

Problem 2.
1. The surface S is the graph of the function $h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$h(x, y) = \tfrac{1}{2}ax^2 + \tfrac{1}{2}by^2.$$

Proceeding as in Example 5 on pages 162–163 (for graphs of functions) we find that the Gaussian curvature at the point $(x, y, h(x, y))$ of S is given by

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2} = \frac{ab}{(1 + a^2x^2 + b^2y^2)^2}.$$

2. Let the plane V be given by the equation $cx + dy = 0$, with $(c, d) \neq (0, 0)$. Using the parametrization $\mathbf{x}(u, v) = (u, v, h(u, v))$ of S as in Example 5, we see that $S \cap V$ is given by $\alpha(t) = \mathbf{x}(u(t), v(t))$, with

$$u(t) = dt, \quad v(t) = -ct.$$

This curve is a line of curvature iff it satisfies equation (8) on page 161. To verify when α satisfies this equation, we have to compute the coefficients E, F and G of the first fundamental form, and the coefficients e, f and g of the second fundamental form of S in the parameterization $\mathbf{x}$. A straightforward computation gives

$$E = 1 + a^2u^2, \quad F = abuv, \quad G = 1 + b^2v^2.$$

Using the expressions derived in Example 5 we get

$$e = \frac{a}{\sqrt{1 + a^2u^2 + b^2v^2}}, \quad f = 0, \quad g = \frac{b}{\sqrt{1 + a^2u^2 + b^2v^2}}.$$

Therefore, α is a line of curvature iff

$$0 = \begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{vmatrix}$$

$$= \frac{1}{\sqrt{1+a^2u^2+b^2v^2}} \begin{vmatrix} c^2 & cd & d^2 \\ 1+a^2d^2t^2 & -abcdt^2 & 1+b^2c^2t^2 \\ a & 0 & b \end{vmatrix}$$

$$= \frac{(a-b)cd}{\sqrt{1+(a^2c^2+b^2d^2)t^2}}.$$

Since $a \neq b$, the curve α is a line of curvature iff $c = 0$ or $d = 0$. If $c = 0$, then V is the xz-plane. If $d = 0$, then V is the yz-plane.

Problem 3.

1. (a) Unit speed parametrizations of these parallels give an acceleration vector in the plane of the parallel. The curve is a geodesic iff this acceleration vector is perpendicular to the surface, which is the case for the inner and outer parallels, but not for the upper parallel.

(b) All parallels are lines of curvature (Book, Example 4 on page 161).

(c) A unit-speed curve is asymptotic iff its acceleration vector (which is $\mathbf{t}'$) is everywhere tangent to the surface (equivalently, iff its normal curvature is zero along the curve, so $\langle \mathbf{t}', N\rangle = 0$). Therefore, only the upper parallel is an asymptotic curve of the torus.

2. The geodesic curvature is the size of the tangential component of the acceleration vector (by definition). So the geodesic curvature of the upper parallel is equal to $\frac{1}{a}$.

Problem 4.

1. Since $\alpha' = \mathbf{x}_u$ and $\alpha'' = \mathbf{x}_{uu}$ (with α' shorthand notation for $\alpha'(s)$, and $\mathbf{x}_u$ shorthand notation for $\mathbf{x}_u(s, v_0)$, and so on), we see that

$$\begin{aligned} \langle \alpha'', \mathbf{x}_u\rangle &= \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle = \tfrac{1}{2}E_u = 0, \\ \langle \alpha'', \mathbf{x}_v\rangle &= \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle = F_u - \tfrac{1}{2}E_v = 0. \end{aligned}$$

In other words, $\alpha''(s) \perp T_{\alpha(s)}S$, so α is a geodesic of S.

2. Since $\beta' = v'\mathbf{x}_v$, we see that $|\beta'|^2 = (v')^2 G$. Therefore, $|\beta'|$ is a positive constant iff

$$v' \neq 0 \quad \text{and} \quad 0 = ((v')^2G)' = 2v'v''G + (v')^3 G_v,$$

which is equivalent to

$$v' \neq 0 \quad \text{and} \quad 2v''G + (v')^2 G_v = 0.$$

3. The curve β has constant non-zero speed, so $\langle \beta', \beta'' \rangle = 0$. Since $\beta' = \nu' \mathbf{x}_\nu$ with $\nu' \neq 0$, we see that $\langle \beta'', \mathbf{x}_\nu \rangle = 0$. Therefore, β is a geodesic iff $\langle \beta'', \mathbf{x}_u \rangle = 0$. A short computation shows that

$$\langle \beta'', \mathbf{x}_u \rangle = \langle \nu'' \mathbf{x}_\nu + (\nu')^2 \mathbf{x}_{\nu\nu}, \mathbf{x}_u \rangle = (\nu')^2 (F_\nu - \tfrac{1}{2} G_u) = -\tfrac{1}{2} G_u (\nu')^2. \qquad (1)$$

Now assume β is a geodesic for every u_0. We have to show that G_u is identically zero. To see this, let (u_0, ν_0) be an arbitrary point in U, and consider the geodesic $\beta(s) = \mathbf{x}(u_0, \nu(s))$, with $\nu(0) = \nu_0$. Since $\langle \beta''(0), \mathbf{x}_\nu(u_0, \nu_0) \rangle = 0$, identity (1) implies $G_u(u_0, \nu_0) = 0$.

Conversely, assume that G_u is identically zero. A similar argument shows that β'' is perpendicular to $\mathbf{x}_u$ along the curve β. According to (1), the acceleration vector β'' is also perpendicular to $\mathbf{x}_u$, so β is a geodesic.